

Normal holomorphic curves from parabolic regions to projective spaces

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Abstract

A holomorphic map $\mathbf{C} \rightarrow \mathbf{P}^n$ is called normal if it is uniformly continuous from the Euclidean metric to the Fubini–Study metric. The paper contains a survey of known results about such maps, as well as some new theorems.

1 Holomorphic curves in projective spaces

This text was written in 1998 as an answer to a question asked by Misha Gromov. As he soon answered this question himself [6], this preprint was not intended for publication. However the recent activity in the subject [25, 26] suggests that the survey part of this paper, which occupies most of it, might be of some use to the researchers in this area. The result which was new in 1998 is in the Appendix, and a stronger result is available now, [25, Thm. 1.5].

We consider holomorphic maps $f : G \rightarrow \mathbf{P}^n$, where G is a region in the complex line \mathbf{C} , and \mathbf{P}^n is the complex projective space of dimension n . Such maps are called holomorphic curves.

We denote by $\Pi : \mathbf{C}^{n+1} \setminus \{0\} \rightarrow \mathbf{P}^n$ the standard projection map. If $w = \Pi(\zeta)$ and $\zeta = (\zeta_0, \dots, \zeta_n)$ we write $w = (\zeta_0 : \dots : \zeta_n)$ (with columns) and call ζ_j homogeneous coordinates of w . The Fubini–Study metric is given in homogeneous coordinates by

$$ds^2 = \frac{\langle d\zeta, d\zeta \rangle \langle \zeta, \zeta \rangle - |\langle \zeta, d\zeta \rangle|^2}{\langle \zeta, \zeta \rangle^2}, \quad (1)$$

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where $\langle ., . \rangle$ stands for the standard Hermitian product in \mathbf{C}^{n+1} . The Fubini–Study distance between two points $\Pi(\zeta)$ and $\Pi(\eta)$ is equal to the “angle” between two complex one-dimensional subspaces passing through ζ and η that is

$$\text{dist}(\Pi(\zeta), \Pi(\eta)) = \arccos \frac{|\langle \zeta, \eta \rangle|}{\|\zeta\| \|\eta\|}, \quad \text{where} \quad \|\zeta\| := \sqrt{\langle \zeta, \zeta \rangle} \quad (2)$$

is the Euclidean norm in \mathbf{C}^{n+1} . So, for example, the diameter of \mathbf{P}^n is equal to $\pi/2$. In what follows all metric notions in \mathbf{P}^n will refer to the Fubini–Study metric, and in \mathbf{C}^k to the Euclidean metric. We use dist and $B(a, r)$ for distances and open balls in both cases, in addition the Euclidean distance will sometimes be written using the norm notation $\|a - b\|$.

Every holomorphic curve can be factored as $f = \Pi \circ \tilde{f}$ where $\tilde{f} : G \rightarrow \mathbf{C}^{n+1} \setminus \{0\}$ is a holomorphic map, called a *reduced homogeneous representation* of f . Thus $\tilde{f} = (f_0, \dots, f_n)$, where f_j are holomorphic functions without common zeros. For a given curve f its reduced homogeneous representation is defined up to multiplication of all coordinates by the same holomorphic function without zeros. We will also use *meromorphic homogeneous representations*, where the homogeneous coordinates are allowed to be meromorphic functions and to have common zeros. Every $(n+1)$ -tuple of meromorphic functions (f_0, f_1, \dots, f_n) in $G \subset \mathbf{C}$ defines a holomorphic curve $f : G \rightarrow \mathbf{P}^n$, except when all functions f_j are identically equal to 0. Indeed, we can always multiply all coordinates by a meromorphic function in G to cancel all common zeros and all poles. *Unless otherwise is explicitly stated we use everywhere only reduced homogeneous representations.* If $n = 1$ the curve $f = (f_0 : f_1)$ is identified with the meromorphic function $f = f_1/f_0 : \mathbf{C} \rightarrow \bar{\mathbf{C}} = \mathbf{P}^1$. Here $\bar{\mathbf{C}}$ is the Riemann sphere; the Fubini–Study metric for $n = 1$ is the spherical metric of constant curvature 4.

The length distortion of a holomorphic curve f (from the Euclidean to the Fubini–Study metric) is described by the *spherical derivative* $f^\#$, and the area distortion by its square. The following explicit expression can be derived from (1):

$$(f^\#)^2 := \frac{\sum_{i < j} |f'_i f_j - f_i f'_j|^2}{\|\tilde{f}\|^4}. \quad (3)$$

Here $\tilde{f} = (f_1, \dots, f_n)$ is a reduced homogeneous representation ($f^\#$ does not depend on its choice). When $n = 1$ we have

$$f^\# = \frac{|f'|}{1 + |f|^2}.$$

An introduction to Nevanlinna–Cartan theory is [14]. The Nevanlinna–Cartan characteristic of a holomorphic curve $f : \mathbf{C} \rightarrow \mathbf{P}^n$ is defined by

$$T(r, f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \|\tilde{f}(re^{i\theta})\| d\theta - \log \|\tilde{f}(0)\|, \quad (4)$$

where \tilde{f} is a reduced homogeneous representation. It is easy to see that T does not depend on the choice of representation. The Laplacian of the subharmonic function $\log \|\tilde{f}\|$ has density $2(f^\#)^2$ with respect to the Lebesgue measure \boxed{dz} in the plane, so by Jensen’s Formula we have

$$T(r, f) = \int_0^r A(t, f) \frac{dt}{t}, \quad \text{where} \quad A(t, f) = \frac{1}{\pi} \int_{|z| \leq t} (f^\#)^2(z) \boxed{dz}. \quad (5)$$

If $n = 1$ this is called the Ahlfors–Shimizu form of the Nevanlinna characteristic, and $A(r, f)$ Ahlfors’ (non-integrated) characteristic. It is equal to the area of the disc $B(0, t) := \{z : |z| \leq t\}$ with respect to the pull-back of the spherical metric, divided by π . As the total area of the Riemann sphere is equal to π , the non-integrated characteristic $A(t, f)$ can be interpreted as the average covering degree of $f : B(0, t) \rightarrow \bar{\mathbf{C}}$.

The order of a curve $f : \mathbf{C} \rightarrow \mathbf{P}^n$ is defined by

$$\rho_f := \limsup_{r \rightarrow \infty} \frac{\log T(r, f)}{\log r}.$$

If f is a curve of finite order ρ , there exists a reduced homogeneous representation whose coordinates have order at most ρ .

If $f : \mathbf{C}^* \rightarrow \mathbf{P}^n$ then the definitions of characteristics have to be slightly modified. We put

$$A(r, f) = \frac{1}{\pi} \int_{\{z : 0 \leq \log |z| / \log r \leq 1\}} (f^\#)^2(z) \boxed{dz}, \quad r > 0$$

and

$$T(r, f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \|\tilde{f}(re^{i\theta})\| d\theta - \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \|\tilde{f}(e^{i\theta})\| d\theta.$$

Then we have again the first relation in (5). There are two values of the order now: $\rho_f(0)$ and $\rho_f(\infty)$, one for each singularity.

2 Normal curves in parabolic regions

The set of all holomorphic curves $G \rightarrow \mathbf{P}^n$, equipped with topology of uniform convergence on compacts in G with respect to the Fubini–Study

metric, forms a complete metric space. A set of holomorphic curves in a region $G \in \mathbf{C}$ is called a *normal family* if the closure of this set is compact. A necessary and sufficient condition for normality is that the family is equicontinuous on every compact subset of G (Ascoli–Arzela Theorem). An equivalent way to say this is that spherical derivatives are uniformly bounded on compacts in G .

Every region $G \in \bar{\mathbf{C}}$ has a complete Riemannian metric of constant curvature, compatible with the conformal structure. For a given region such metric is defined up to a constant multiple. We choose the following normalizations. In $\bar{\mathbf{C}}$ we take the spherical metric defined above, it has curvature 4. In \mathbf{C} we choose the standard Euclidean metric and in \mathbf{C}^* the Riemannian metric $|dz|/|z|$, both of them of zero curvature. In a hyperbolic region there exists unique complete conformal metric of curvature -4 which comes from the metric $|dz|/(1-|z|^2)$ in the unit disc \mathbf{U} via the Uniformization Theorem. We call these metrics intrinsic for G . The group of isometries is denoted by $\text{Iso}(G)$. There are four regions, namely $\bar{\mathbf{C}}$, \mathbf{C} , \mathbf{C}^* and \mathbf{U} where the group of isometries acts transitively.

Let G be a region whose group of isometries acts transitively. A holomorphic curve $f : G \rightarrow \mathbf{P}^n$ is called *normal* if it satisfies the following equivalent conditions

1. The family $\{f \circ \phi : \phi \in \text{Iso}(G)\}$ is normal.
2. f is uniformly continuous from the intrinsic metric of G to the Fubini–Study metric.
3. $\sup_{z \in G} f^\#(z)/\rho(z) < \infty$, where ρ is the ratio of the intrinsic metric to the Euclidean metric.

□

We reserve the name *normal function* for the case $n = 1$. The set of normal curves will be denoted by $\mathfrak{Y}_{G,n}$, or \mathfrak{Y}_n if $G = \mathbf{C}$. For $K > 0$ we set $\mathfrak{Y}_{G,n}(K) = \{f \in \mathfrak{Y}_{G,n} : \sup(f^\#/\rho) \leq K\}$. For every G, K and n the set $\mathfrak{Y}_{G,n}(K)$ is compact and the group $\text{Iso}(G)$ acts on it by translations: $f \mapsto \phi f = f \circ \phi^{-1}$, $\phi \in \text{Iso}(G)$.

The subject of this paper is normal holomorphic curves defined in parabolic regions \mathbf{C} and \mathbf{C}^* . The elements of \mathfrak{Y}_1 are called sometimes Yosida functions. They were introduced by Julia [13] and studied by Yosida [27]. The importance of the class \mathfrak{Y}_n is partially explained by the following theorem, based on the idea of Lohwater and Pommerenke [17] (case $n = 1$). The same

idea was used effectively by Brody [14, Ch. III], for holomorphic curves to compact manifolds. See also [28, 18].

Theorem 2.1 *Let M be a set of holomorphic curves in \mathbf{C} , containing non-constant curves and having the following properties:*

- (i) *if $f \in M$ and $L(z) = az + b$, $a \neq 0$, then $f \circ L \in M$;*
- (ii) *$M \cup \{\text{constant curves}\}$ is closed.*

Then M contains non-constant normal curves.

This theorem is useful because in some cases it permits to reduce Picard-type theorems to their special cases for curves in \mathfrak{Y}_n . The requirement (i) can be substantially relaxed [22, 23]. Examples of applications are in [1, 2, 3, 9]; the survey of related results in dimension 1 is [29].

Proof of Theorem 2.1. Let $f \in M$ be a non-constant curve. Put

$$M_n := \max_{|z| \leq n} (n - |z|) f^\#(z) := (n - |z_n|) f^\#(z_n), \quad |z_n| < n.$$

Evidently $M_n \rightarrow \infty$. So

$$\rho_n := \frac{1}{f^\#(z_n)} = \frac{n - |z_n|}{M_n} = o(n - |z_n|). \quad (6)$$

We put $g_n(z) = f(z_n + \rho_n z)$. Then

$$g_n^\#(0) = \rho_n f^\#(z_n) = 1, \quad (7)$$

and for any fixed $r > 0$ and $|z| \leq r$ we have, using (6) and the definition of M_n :

$$\begin{aligned} g_n^\#(z) &= \rho_n f^\#(z_n + \rho_n z) \\ &\leq \frac{n - |z_n|}{M_n} \max_{|z| \leq |z_n| + \rho_n r} f^\#(z) \\ &\leq \frac{n - |z_n|}{n - |z_n| - \rho_n r} \cdot \frac{1}{M_n} \cdot \max_{|z| \leq |z_n| + \rho_n r} (n - |z|) f^\#(z) \\ &\leq (1 + o(1)) \frac{1}{M_n} \max_{|z| \leq n} (n - |z|) f^\#(z) = (1 + o(1)). \end{aligned}$$

Thus $\{g_n\}$ is a normal family, and we can choose a subsequence such that $g_n \rightarrow g$, where g is non-constant holomorphic curve in view of (7). The assumption (i) implies that $g_n \in M$, and thus by (ii) we have $g \in M$. We also have $g^\# \leq 1$, so g is normal. \square

It follows from (5) that $f \in \mathfrak{Y}_n(K)$ satisfy

$$T(r, f) \leq K^2 r^2 / 2, \quad r > 0, \quad (8)$$

so they are of order at most 2, normal type.

The following characterization of \mathfrak{Y}_n belongs to Montel and Yosida for $n = 1$. A set of hypersurfaces in \mathbf{P}^n is called admissible if every $n + 1$ hypersurfaces of this set have empty intersection.

Theorem 2.2 *Let H_1, \dots, H_{3n+1} be an admissible set of hypersurfaces and f a holomorphic curve. Denote by $E_j = f^{-1}(H_j)$ the preimages of these hypersurfaces. Then $f \in \mathfrak{Y}_n$ if and only if the following condition is satisfied: there exists $\delta > 0$ such that every disc of diameter δ in \mathbf{C} intersects at most n of the sets E_j .*

Remark. For $n = 1$ we need 4 points (any set of points in $\bar{\mathbf{C}}$ is admissible). There are examples showing that three points may not be enough.

Proof of Theorem 2.2. Let $f \in \mathfrak{Y}_n(K)$, and $B \subset \mathbf{C}$ a disc of diameter δ . Then $\text{diam} f(B) \leq K\delta$, so if δ is small enough, $f(B)$ cannot intersect $n + 1$ hypersurfaces. Otherwise there would be a sequence of balls $B(w_k, r_k) \subset \mathbf{P}^n$ with $r_k \rightarrow 0$, each ball intersecting $n + 1$ hypersurfaces. By passing to a subsequence we may assume that these $n + 1$ hypersurfaces are the same for all balls, say H_1, \dots, H_{n+1} . We can also assume that $w_k \rightarrow w \in \mathbf{P}^n$. But then $w \in H_1 \cap \dots \cap H_{n+1}$ and this contradicts our assumption that the system of hypersurfaces is admissible. This proves “only if” part of Theorem 2.

Now we assume that for some $\delta > 0$ every disc of diameter δ intersects at most n of E_j . Fix such a disc B and notice that at least $2n + 1$ hypersurfaces are omitted in B . It remains to use the following generalization of Landau’s theorem: *if a holomorphic curve in a disc omits $2n + 1$ hypersurfaces from an admissible system then its spherical derivative is bounded on every compact in this disc by a constant, depending only on the hypersurfaces and the compact* [11, 12]. \square

Theorem 2.3 *Let $f = (f_0 : \dots : f_n)$ be a holomorphic curve. If all ratios f_i/f_j belong to $\mathfrak{Y}_1(K)$ then $f \in \mathfrak{Y}_n(K\sqrt{n})$.*

The converse is not true as the following example shows: $f(z) = (\cos z : \cos(\alpha z) : z)$, where $\alpha \in (0, 1)$ is irrational. To show that $f \in \mathfrak{Y}_2$ consider a sequence $\lambda_k \rightarrow \infty$. By choosing a subsequence we may assume that one of the following cases holds:

- (i) $(\cos \lambda_k)/\lambda_k \rightarrow \infty$,
- (ii) $(\cos \lambda_k)/\lambda_k \rightarrow 0$, or
- (iii) $(\cos \lambda_k)/\lambda_k \rightarrow a \in \mathbf{C}^*$.

It is easy to see that the translations $t_{\lambda_k} f$ converge to $(1 : 0 : 0)$, $(0 : 0 : 1)$ or $(a \exp(\pm 2iz) : 0 : 1)$ in cases (i), (ii) and (iii) respectively. So $f \in \mathfrak{Y}_2$. On the other hand, meromorphic function $\cos z / \cos(\alpha z)$ does not belong to \mathfrak{Y}_1 because some of its zeros are very close to poles, which cannot happen for a uniformly continuous function.

Proof of Theorem 2.3. If the spherical derivatives of all ratios are at most K , we have

$$\sum_{i < j} |f'_i f_j - f_i f'_j|^2 \leq K^2 \sum_{i < j} (|f_i|^2 + |f_j|^2)^2 \leq K^2 n(|f_1|^2 + \dots + |f_n|^2)^2.$$

□

The following results about free interpolation for Yosida curves were stated by M. Gromov in his lecture in Tel Aviv University in November 1997.

A set $E \in \mathbf{C}$ is called *K-sparse* if the distance between points of E is at least K .

Theorem 2.4 *There exists $C(n) > 0$, depending only on dimension, such that every function $E \rightarrow \mathbf{P}^n$, defined on an K -sparse set E , can be interpolated by a $f \in \mathfrak{Y}_n(C(n)/K)$.*

The proof is given in the Appendix. It is not clear whether a similar result is true with a constant C independent on dimension.

A set $E \in \mathbf{C}$ is called *K-dense* if every square with side length K contains at least one point of E .

Theorem 2.5 *For every K -dense set E and every f_0 and f_1 in $\mathfrak{Y}_n(c/K)$ with $c < \sqrt{\pi/2}$ the equality $f_0|_E = f_1|_E$ implies $f_0 = f_1$.*

Proof of Theorem 2.5. Let $n_E(r) = \text{card}\{z \in E : |z| \leq r\}$. Then $n_E(r) \geq \pi(r/K)^2 + O(r)$ so, assuming wlog that $0 \notin E$,

$$N_E(r) := \int_0^r n_E(t) \frac{dt}{t} \geq \frac{\pi r^2}{2K^2} + O(r). \quad (9)$$

On the other hand the assumptions of Theorem 2.5 and (8) imply

$$T(r, f_j) \leq (cr)^2 / (2K^2), \quad j = 0, 1.$$

Now take a fractional-linear function $L : \mathbf{P}^n \rightarrow \bar{\mathbf{C}}$, and put $g_j := L \circ f_j$, $j = 0, 1$. Then we have $T(r, g_j) \leq T(r, f_j) + O(1)$ and $T(r, g_0 - g_1) \leq T(r, g_0) + T(r, g_1) + \log 2$. Thus $T(r, g_0 - g_1) \leq (cr/K)^2 + O(1)$. Now (9) implies that $N(r, 0, g_0 - g_1) \geq \pi r^2/(2K^2) + O(r)$, so we obtain from the First Main Theorem of Nevanlinna that $g_0 = g_1$. As this conclusion is valid for every fractional linear function L we conclude that $f_0 = f_1$. \square

3 Normal functions in \mathbf{C}^*

In this section we consider the class $\mathfrak{Y}_{\mathbf{C}^*, n}$ consisting of holomorphic curves $\mathbf{C}^* \rightarrow \mathbf{P}^n$, with $\sup |z|f^\#(z) < \infty$. It is naturally isomorphic to the subclass of $2\pi i$ -periodic curves in $\mathfrak{Y}_{\mathbf{C}, n}$. The multiplicative group of \mathbf{C}^* acts on \mathbf{C}^* by isometries $z \mapsto \lambda z$, $\lambda \in \mathbf{C}^*$. So our curves $f : \mathbf{C}^* \rightarrow \mathbf{P}^n$ are characterized by the property that the families of their translations $\{h_\lambda : \lambda \in \mathbf{C}^*\}$, where $h_\lambda f(z) := f(\lambda z)$, are normal. The class $\mathfrak{Y}_{\mathbf{C}^*, 1}$ will be also called \mathcal{O}_1 after A. M. Ostrowski, who gave in [20] an explicit parametric description of the class (see Theorem 3.1 below). In fact Ostrowski considered slightly smaller class of functions which have no essential singularity at 0. This subclass was introduced by Julia in connection with the so-called Julia directions. Ostrowski functions with no essential singularity at 0 are exactly meromorphic functions in \mathbf{C} which have no Julia directions. This subclass was studied by Julia, Montel and Ostrowski under the name of “exceptional functions”. (“Exceptional”, because they have no Julia directions). Chapter VI of [19] contains a detailed exposition of this work, including the remarkably complete result of Ostrowski, which we slightly generalize here in

Theorem 3.1 *A meromorphic function $f : \mathbf{C}^* \rightarrow \bar{\mathbf{C}}$ belongs to $\mathcal{O}_1(K)$ if and only if it admits a representation*

$$f(z) = az^m \frac{\prod_{k \geq 0} \left(1 - \frac{z}{a_k}\right) \prod_{k < 0} \left(1 - \frac{a_k}{z}\right)}{\prod_{k \geq 0} \left(1 - \frac{z}{b_k}\right) \prod_{k < 0} \left(1 - \frac{b_k}{z}\right)}, \quad (10)$$

where $a \in \mathbf{C}$, $m \in \mathbf{Z}$, $a_k \in \mathbf{C}^*$, $b_k \in \mathbf{C}^*$, both sequences (a_k) and (b_k) may be finite or infinite in one or both directions, they tend to 0 as $k \rightarrow -\infty$, and they tend to ∞ as $k \rightarrow +\infty$, and the following four conditions are satisfied:

- (i) *The number of zeros a_k and poles b_k of f in every ring of the form $\{z : r < |z| < 2r\}$, $r > 0$ (counting multiplicity) is bounded by a constant $C_1(K)$.*

(ii) the difference between the number of zeros and poles of f in every ring $\{z : r_1 < |z| < r_2\}$, $r_1 > 0$, $r_2 > 0$ (counting multiplicity) is bounded by a constant $C_2(K)$.

(iii) For every p and q the ratios

$$|a_p|^m \frac{\prod_{k: 0 \leq \log |a_k| / \log |a_p| \leq 1} \frac{|a_p|}{|a_k|}}{\prod_{k: 0 \leq \log |b_k| / \log |a_p| \leq 1} \frac{|a_p|}{|b_k|}} \quad \text{and} \quad |b_q|^m \frac{\prod_{k: 0 \leq \log |b_k| / \log |b_q| \leq 1} \frac{|b_q|}{|b_k|}}{\prod_{k: 0 \leq \log |a_k| / \log |b_q| \leq 1} \frac{|b_q|}{|a_k|}}$$

are bounded from above by a constant $C_3(K)$.

(iv) For every pair (k, j) the distance between a_k and b_j is bounded away from zero by a positive constant $C_4(K)$.

There is a simple geometric interpretation of conditions (i)-(iii), also given in [20]. The Jensen's formula for meromorphic functions in a ring $A(r_1, r_2)$ has the form

$$\begin{aligned} & \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |f(r_2 e^{i\theta})| d\theta - \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |f(r_1 e^{i\theta})| d\theta \\ &= \int_{r_1}^{r_2} (n(t, 0) - n(t, \infty)) d(\log t) + s(\log r_2 - \log r_1), \end{aligned}$$

where s is an integer. Now for a function f of the form (10) we put

$$\phi(t) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |f(e^{t+i\theta})| d\theta.$$

Then ϕ is a piecewise linear function on \mathbf{R} . The jumps of derivative correspond to zeros and poles of f . Namely, the jump of derivative $\phi'_+(t) - \phi'_-(t)$ is equal to the number of zeros minus the number of poles on the circle $\{z : |z| = \exp(t)\}$. So each time the derivative jumps by an integer. Condition (i) implies that all jumps have bounded magnitude and the number of jumps on any interval of length $\log 2$ is bounded. Condition (ii) means that the algebraic sum of jumps on any interval is bounded. This is equivalent to the boundedness of ϕ' on the whole real line. Finally, condition (iii) means the following: there is a horizontal strip such that whenever the graph of ϕ is above the strip, ϕ is concave, and whenever the graph is below the strip, ϕ is convex. Let us call piecewise linear functions with such properties *admissible*.

Once an admissible piecewise linear function is given, one can construct function $f \in \mathcal{O}_1$ by prescribing the arguments of zeros and poles at each point of jump of ϕ' , such that condition (iv) is satisfied.

Thus Theorem 3.1 gives a simple and effective parametric description of the class \mathcal{O}_1 in terms of their zeros, poles and constants a and m .

Using (i) one can improve Theorem 2.2 for the case of functions $\mathbf{C}^* \rightarrow \bar{\mathbf{C}}$: *Let f be a meromorphic function in \mathbf{C}^* and $E_j = f^{-1}(a_j)$, $j = 1, 2, 3$. Then for $f \in \mathcal{O}_1$ it is necessary and sufficient that for some $\delta > 0$ every disc of diameter δ in \mathbf{C}^* intersects at most one of the sets E_j .* See [19, p. 162]

It is interesting that for curves $\mathbf{C}^* \rightarrow \mathbf{P}^n$ there is a universal lower bound for $\sup |z|f^\#(z)$. This was discovered by Lehto and Virtanen [16] who used a geometric method; then Lehto in [15] published a very simple analytic proof with precise constant (for $n = 1$). We follow the method of Lehto.

Theorem 3.2 *For every non-constant holomorphic curve $f : \mathbf{C}^* \rightarrow \mathbf{P}^n$ we have*

$$\sup_z |z|f^\#(z) \geq 1/2.$$

So $\mathcal{O}_1(K)$ consists of constants for $K < 1/2$.

Proof. There are two cases to consider.

Case 1. The order of f at one of the singularities, 0 or ∞ is positive. Then the statement of the theorem follows immediately from (5).

Case 2. The order at both singularities is zero. Let $f = (f_0 : \dots : f_n)$ be a reduced representation such that all f_j are functions of zero order at both singularities. For every $w \in \mathbf{C}^*$ consider the function

$$g(z, w) := f_0(z)\overline{f_0(\overline{zw})} + \dots + f_n(z)\overline{f_n(\overline{zw})}.$$

Then g is a holomorphic function in $\mathbf{C}^* \times \mathbf{C}^*$. For fixed w the function $g_w : z \rightarrow g(z, w)$ has zero order at both singularities, so it either has zeros in \mathbf{C}^* or has the form

$$g_w(z) = h(w)z^q, \tag{11}$$

where q is an integer. If for some w on the unit circle g_w has a zero $z^* \in \mathbf{C}^*$ then the points $(f_0(z^*) : \dots : f_n(z^*))$ and $(f_0(\overline{z^*w}) : \dots : f_n(\overline{z^*w}))$ are “diametrically opposite points” in \mathbf{P}^n that is the distance between them is equal to $\pi/2$. In this case f assumes two diametrically opposite values on the circle $|z| = |z^*|$. The intrinsic length of this circle is equal to 2π , so it follows that there is a point z on this circle where $|z|f^\#(z) \geq 1/2$.

It remains to consider the possibility that (11) holds for all w on the unit circle and all $z \in \mathbf{C}^*$. In this case (11) actually holds for all z and w in

\mathbf{C}^* . We examine this possibility by substituting for \tilde{f} a Laurent series with undetermined coefficients:

$$f_j(z) = \sum_{k=-\infty}^{\infty} c_{j,k} z^k.$$

The resulting system of equations shows that the functional equation (11) has no solutions for which the curve f is not constant. \square

Remarks.

1. As curves $f \in \mathcal{O}_1$ have zero order, the second part of the previous proof applies to them. It shows that for such curve f there always exist two points in the same circle $\{z : |z| = r\}$ whose images are diametrically opposite. It is not clear whether such improvement of Theorem 3.2 is true for curves of order ≥ 1 .

2. The function $f(z) = z : \mathbf{C}^* \rightarrow \bar{\mathbf{C}}$ shows that the estimate $1/2$ in Theorem 3.2 is precise. Probably the only functions for which $\sup |z|f^\#(z) = 1/2$ are $f(z) = kz$ with $k \in \mathbf{C}^*$. The following example from [15] shows that for every $\epsilon > 0$ there is a “periodic” function $f : \mathbf{C}^* \rightarrow \bar{\mathbf{C}}$, that is $f(tz) = f(z)$ for some $t > 1$ and such that $\sup |z|f^\#(z) \leq 1/2 + \epsilon$. Put

$$f(z) = \prod_{k \in \mathbf{Z}} \frac{z + t^k}{z - t^k}.$$

Then direct computation shows that $\max_{\mathbf{C}^*} |z|f^\#(z) \rightarrow 1/2$ as $t \rightarrow \infty$. This implies that there is an open discrete map from a flat torus whose shortest closed geodesic has length 2π to the sphere whose great circles have length π , such that the length distortion is arbitrarily close to $1/2$. That a continuous map of non-zero degree with such properties exists follows from Proposition 2.12 in [5].

3. The proof in Case 2 shows that actually some circle $|z| = r$ has image, not shorter than a great circle. It is interesting to consider the special case when f has no singularities, that is extends to the whole sphere. For this case Theorem 3.2 follows from

Proposition 3.3 *Let $f : \bar{\mathbf{C}} \rightarrow \mathbf{P}^n$ be a continuous map of non-zero degree. Then some circle $|z| = r$ has image of length at least π .*

This can be proved in the same way as Proposition 2.12 in [5].

Our proof of the existence of free interpolation in the Appendix shows that for periodic interpolation data one can find periodic interpolating function. Thus Theorems 2.4 and 2.5 have counterparts for the class \mathcal{O}_n .

4 Binormal curves

A holomorphic curve $f \in \mathfrak{Y}_{G,n}$ is called *binormal* if the family of its translations by the elements of $\text{Iso}(G)$ is normal and has *no constant limit curves*. Yosida in [27] called them functions of first category. Notice that the closure of the set of translations of a binormal curve consists of only binormal curves. The following characterization was given by Yosida in [27] for $n = 1$.

Theorem 4.1 *A curve $f \in \mathfrak{Y}_n$ is binormal if and only if for every $\delta > 0$ there exists a constant c such that*

$$\int_{|z-\zeta|<\delta} (f^\#)^2(z) \boxed{dz} \geq c \quad \text{for every } \zeta \in \mathbf{C}.$$

□

Corollary 4.2 *Binormal curves f satisfy*

$$T(r, f) \asymp r^2.$$

□ The idea of the following result is contained in [27], but the result is stated there in a weaker form, and Yosida's proof of it contains mistakes (his Lemma 1 is incorrect). We use the standard notations of Nevanlinna theory, in particular, for the averaged counting function N_1 of critical points

Theorem 4.3 *For binormal functions $f \in \mathfrak{Y}_1$ we have*

$$N_1(r, f) = 2T(r, f) + O(1) \tag{12}$$

and

$$N(r, a, f) = T(r, f) + O(1). \tag{13}$$

Both statements follow from

Proposition 4.4 *Let f be a binormal function. If the arc α is the intersection of a circle of radius greater than 1 with a disc of radius 1, then*

$$\int_{\alpha} |\log f^\#(z)| |dz| \leq c \tag{14}$$

and for every $a \in \bar{\mathbf{C}}$

$$\int_{\alpha} |\log (\text{dist}(f(z), a))^{-1}| |dz| \leq c \tag{15}$$

where c is a constant depending only on f .

See [27, Lemma 2] or [2]. Now (13) is an immediate consequence from (15) and the First Main Theorem of Nevanlinna, and (12) can be derived from (14) as in [2]. It follows that the error term in the Second Main Theorem of Nevanlinna is bounded for binormal functions f . It is also proved in [2] that one can “differentiate” the asymptotic relations (12) and (13). More precisely,

$$n_1(r, f) = (2 + o(1))A(r, f), \quad r \notin E,$$

and

$$n(r, a, f) = (1 + o(1))(A(r, f)), \quad r \notin E,$$

where A is the (non-integrated) Ahlfors’ characteristic, n_1 is the number of critical points in $B(0, r)$, and the exceptional set E has zero density.

We have the following corollaries, all of them due to Yosida.

Corollary 4.5 *If $f \in \mathfrak{Y}_1$ is binormal then it has no deficiencies, and even no Valiron deficiencies. This implies that f assumes every value in $\bar{\mathbf{C}}$. \square*

Corollary 4.6 *If $f \in \mathfrak{Y}_1$ is binormal then preimage of every point is K -dense in \mathbf{C} with some $K > 0$.*

Indeed if, say poles, are not dense there is a sequence of pole-free squares with centers λ_k and side length tending to infinity. As the family $t_{\lambda_k}f$ is normal, we can choose a subsequence converging to g , where g is also binormal. On the other hand g has no poles at all, which contradicts Corollary 4.5. \square

A property which follows immediately from the definition should be also mentioned: *binormal functions have no asymptotic values*.

Thus binormal functions display asymptotic behavior, similar to elliptic functions. In certain sense such behavior (described in preceding corollaries) is “typical” for meromorphic functions. In the next section we consider the “opposite extreme” to (12).

Using Theorem 3.1 we can give explicit description of binormal functions in \mathcal{O}_1 , that are meromorphic functions f in \mathbf{C}^* such that the families $\{h_\lambda f : \lambda \in \mathbf{C}^*\}$ are normal without constant limit functions.

Theorem 4.7 *A meromorphic function $f \in \mathcal{O}_1$ is binormal if and only if in representation (10) the following additional property is satisfied:*

(v) *There exists $C_5(f) > 0$ such that every annulus $\{z : r < |z| < c_5 r\}$, $r > 0$ contains at least one zero and at least one pole of f .*

The additional property (v) implies that all ratios in (iii) are also bounded away from zero or, which is equivalent, the piecewise-linear function ϕ introduced after Theorem 3.1 is bounded.

Proof of Theorem 4.7. The necessity of condition (v) follows from Corollary 4.6, applied to the binormal function $f \circ \exp \in \mathfrak{Y}_1$. Sufficiency is evident, because if condition (v) is satisfied, all limit functions have zeros and poles. \square

5 Locally univalent and entire normal functions in \mathbf{C}

Theorem 5.1 *The only locally univalent functions in \mathfrak{Y}_1 are exponential functions and fractional-linear functions. By exponential we mean $L \circ \exp(az)$, where L is a fractional linear transformation and $a \in \mathbf{C}^*$.*

This is proved in [2]. Using [10] one can probably relax the assumption of local univalence in this theorem, by replacing it with the weaker assumption $N_1(r, f) = o(T(r, f))$, and still preserve the conclusion.

It follows from a theorem of Clunie and Hayman [4] that for entire functions in \mathfrak{Y}_1 the growth estimate (8) can be substantially improved. We give a refined version of this theorem based on the work of Pommerenke [21] and Minda [18].

Theorem 5.2 *For an entire function f the following conditions are equivalent:*

- (i) $\sup f^\# \leq 1$.
- (ii) $|f(z)| \leq 1$ implies $|f'(z)| \leq 2$.
- (iii) $\nabla w \leq 2$, where $w := \log^+ |f|$.

Remarks. It follows from (iii) that $w(z) \leq 2|z|$ that is

$$|f(z)| \leq \exp \max\{2|z|, 1\},$$

so f has at most exponential type.

Let E be a subset of \mathbf{C} . If $|E| \geq 5$ then the condition that $f^\#$ is bounded on $f^{-1}(E)$ is equivalent to $f \in \mathfrak{Y}_1$ for meromorphic function f . In the case of entire functions $|E| \geq 3$ is enough (see [8] for these results). Thus the condition (ii) in Theorem 5.2 can be replaced by much weaker condition.

Proof. The implication (i) \Rightarrow (ii) and (iii) \Rightarrow (i) are evident.

To prove (ii) \Rightarrow (iii) we set $D = \{z : |f(z)| > 1\}$, and for every $R > 0$ consider the following function in D :

$$u_R = \frac{|f'|}{|f|(\log |f| + R)}.$$

Evidently

$$u_R(z) \leq 2/R, \quad z \in \partial D, \quad (16)$$

and u_R satisfies

$$\Delta \log u \geq u^2 \quad (17)$$

in the sense of distributions (by direct verification).

Proposition 5.3 *If u is a positive continuous function in a plane domain D with the properties (16) and (17) then $u \leq 2/R$ in D .*

Proof of Proposition 5.3. Assume that our Proposition is not true, so $u(z_0) > 2/R$ for some $z_0 \in D$. Consider the function

$$v = \frac{2R}{R^2 - |z - z_0|^2}, \quad z \in B(z_0, R)$$

Evidently

$$v(z) \geq 2/R, \quad z \in B(z_0, R) \quad (18)$$

and (by direct verification)

$$\Delta \log v = v^2. \quad (19)$$

Consider the set

$$K = \{z \in D \cap B(z_0, R) : u(z) > v(z)\}.$$

We have by assumption $u(z_0) > 2/R = v(z_0)$ so $z_0 \in K$. Let D_0 be the component of z_0 in K . Then we have

$$u(z) = v(z), \quad z \in \partial D_0$$

because the inequality $u(z) \leq v(z)$ holds for $z \in \partial D$ (because of (16) and (18)), and for $z \in \partial B(z_0, R)$ (because for such z we have $v(z) = +\infty$).

On the other hand by (17) and (19) we have

$$\Delta(\log u - \log v) \geq u^2 - v^2 > 0 \quad \text{in } D_0$$

so $\log u - \log v$ is a positive subharmonic function in D_0 , zero on the boundary. This contradicts Maximum Principle, so the Proposition is proved. \square

Applying Proposition 5.3 to our function u_R we obtain

$$|f'| \leq \frac{2}{R} \max\{|f|, 1\}(\log^+ |f| + R) \quad (20)$$

(in D this follows from Proposition 5.3, in the rest of the plane from (ii). Because this is true for arbitrary $R > 0$ we obtain (by letting $R \rightarrow \infty$ with fixed z):

$$|f'| \leq 2 \max\{|f|, 1\} \quad (21)$$

Now we put $w = \log^+ |f|$, so that $\nabla w(z) = |f'/f|(z)$ when $|f(z)| > 1$ and rewrite the last equation as $\nabla w \leq 2$, which is (iii). \square .

Notice that the auxiliary function u_R used in the proof is the density of the pull-back of the Poincare metric of the punctured disc $\{w : |w| > 1/R\}$, and Proposition (5.3) is a version of the Ahlfors–Schwarz lemma.

Example 5.4 *There are entire functions in \mathfrak{V}_1 of every order $\rho \in [0, 1]$.*

This example is due to A. Fryntov (private communication). If $\rho = 1$ we take $f(z) = \exp(z)$. Now assume that $\rho \in (0, 1)$ put

$$f(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{2^k}\right)^{[2^{k\rho}]},$$

where $[x]$ stands for the greatest integer $\leq x$. Subharmonic uncton $u := \log |f|$ satisfies the approximate functional equation

$$u(2z) = 2^\rho u(z) + u_0(z),$$

where u_0 is negligible. This permits to verify (iii) in Theorem 5.2. To construct an example of order 0 we replace ρ in the previous formula by $\rho(k) := 1/\log^+ k$. \square

6 Appendix. Proof of existence of free interpolation

We start with establishing some notations. By an automorphism of \mathbf{P}^n we mean $\Pi \circ U \circ \Pi^{-1}$, where U is a unitary transformation of \mathbf{C}^{n+1} . So automorphisms are biholomorphic isometries.

Let $B(r) = \{\zeta \in \mathbf{C}^n : \|\zeta\| < r\}$ be the open ball of radius r centered at the origin and $\bar{B}(r)$ its closure. Consider one of the standard local coordinates in \mathbf{P}^n , namely

$$\psi : B(2) \rightarrow \mathbf{P}^n, \quad \psi(\zeta_1, \dots, \zeta_n) = (1 : \zeta_1 : \dots : \zeta_n). \quad (22)$$

The length distortion by ψ is estimated by

$$\frac{1}{1 + \|\zeta\|^2} \leq \frac{ds}{\|d\zeta\|} \leq \frac{1}{\sqrt{1 + \|\zeta\|^2}} \quad (23)$$

which implies

$$\frac{1}{5} \leq \frac{ds}{\|d\zeta\|} \leq 1 \quad \text{for } \zeta \in B(2). \quad (24)$$

So $\psi^{-1} \in \text{Lip}(5)$. Let p_j be the automorphism of \mathbf{P}^n which interchanges the homogeneous coordinate number 0 with the homogeneous coordinate number j , where $j \in \{1, \dots, n\}$. Let $V = B(2) \times \{0, \dots, n\}$, let $p : V \rightarrow B(2)$ be the projection map, and $W = p^{-1}(\bar{B}(1)) \subset V$. We consider V as a (disconnected) Riemannian manifold equipped with the pull-back of the Euclidean metric via p . So the distance between different sheets $B(2) \times \{i\}$ and $B(2) \times \{j\}$, $j \neq i$, is infinite. The following property is evident:

$$\{a \in V : \text{dist}(a, W) \leq \epsilon\} \text{ is a complete metric space if } \epsilon < 1. \quad (25)$$

Evidently p has a continuous right inverse in every ball of radius 1 centered at a point of $\bar{B}(1)$, and this inverse is an isometry onto the image. A point $a \in V$ will be denoted by $a = (p(a); j) = (\zeta; j) = (\zeta_1, \dots, \zeta_n; j)$, where $\zeta \in B(2)$ and $j \in \{0, \dots, n\}$. We use the notation $\|a' - a''\|$ for the distance between two points in V .

We define

$$\Psi : V \rightarrow \mathbf{P}^n, \quad \Psi(a) = \Psi(p(a); j) = p_j \circ \psi \circ p(a).$$

Then Ψ is a surjective local diffeomorphism; in fact

$$\Psi|_W : W \rightarrow \mathbf{P}^n \text{ is surjective.} \quad (26)$$

Indeed, for every point in \mathbf{P}^n one can choose homogeneous coordinates such that one coordinate is equal to 1 and the rest have absolute value at most 1. Moreover,

$$\begin{aligned} &\text{for every point } a \in W \text{ there exists right inverse} \\ &\Psi^{-1} \text{ defined in } B(p(a), \delta) \subset \mathbf{P}^n \text{ with } \delta := 1/11, \\ &\text{such that } \Psi^{-1}(p(a)) = a \text{ and } \Psi^{-1} \in \text{Lip}(5). \end{aligned} \quad (27)$$

Now we can solve a two-point interpolation problem with uniform estimates.

Lemma 6.1 *For every point $q \in \mathbf{P}^n$ there exists a map $g : \mathbf{C} \times V \rightarrow \mathbf{P}^n$, with the following properties:*

- (a) $z \mapsto g(z, a), \mathbf{C} \rightarrow \mathbf{P}^n$ is holomorphic and its spherical derivative is uniformly bounded with respect to a .
- (b) The map $a \mapsto g(0, a) : V \rightarrow \mathbf{P}^n$ has surjective restriction on W and satisfies the condition (27).
- (c) $\text{dist}(g(z, a), q) \leq (25/4)|z|^{-3} < (1/4)$ for $|z| \geq 3$.
- (d) $\text{dist}(g(z, a'), g(z, a'')) \leq (5/4)\|a' - a''\| |z|^{-4}$ for $|z| \geq 3$.

Proof. We take in this proof for convenience $q = (1 : 1 : \dots : 1)$. Then the general case can be obtained by composing g with an appropriate automorphism of \mathbf{P}^n . We will construct g with properties (a), (c), (d) and $g(0, a) = \Psi(a)$, $a \in V$. Then (b) will be satisfied in view of (26) and (27). Let us assume for simplicity that $a = (\zeta_1, \dots, \zeta_n; 0)$. Construction for $a = (\zeta; j)$ is then obtained by composition with p_j , $1 \leq j \leq n$. (Notice that $p_j(q) = q$). We put

$$g(z, a) := (g_0(z, a) : \dots : g_n(z, a)) := (z^4 + 1 : z^4 + 4z + \zeta_1 : z^4 + \zeta_2 : \dots : z^4 + \zeta_n),$$

that is $g_j(z, a) = z^4 + \zeta_j$, $2 \leq j \leq n$. First we notice that for $|z| \geq 3$ the following estimates hold (we use $|\zeta_j| < 2$):

$$|1 - z^{-4}g_j(z, a)| \leq 5|z|^{-3} < 1/5 \quad (28)$$

$$|g_j(z, a') - g_j(z, a'')| \leq \|a' - a''\| \quad (29)$$

Now (c) is easy to verify, using (1) and (28), and (d) follows from (1), (28) and (29).

It remains to prove (a). From (c) follows that $g^\#$ is uniformly bounded with respect to a for $|z| > 4$ (Cauchy estimate for derivatives). Now let $|z| \leq 4$. It is enough to show that the function $(z, a) \rightarrow g^\#(z, a)$ is continuous for $a = (\zeta; 0)$, $(z, \zeta) \in \bar{B}(4) \times \bar{B}(2)$. In view of (3) it is enough to check that the denominator in (3) is never equal to zero for $(z, \zeta) \in \bar{B}(4) \times \bar{B}(2)$, that is g_j never have common zero. In fact we will show that the first two coordinates g_0 and g_1 never have common zero. The only zeros of g_0 are $\pm 1, \pm i$. So our assertion follows from the fact that g_1 never has zeros in the ring $\{z : 3/4 < |z| < 5/4\}$ which follows from Rouché theorem. \square

Functions g interpolate at two points, 0 and ∞ . Now we are going to combine our two-point solutions, essentially by adding them, so we need a surrogate of addition in \mathbf{P}^n . Let

$$\Delta := B(\delta) = B(1/11) \subset \mathbf{C}^n \quad \text{and} \quad q := (1 : 0 : \dots : 0) \in \mathbf{P}^n. \quad (30)$$

These notations will be fixed till the end of the proof of Theorem 2.4. We define a map $P : \mathbf{P}^n \times \Delta \rightarrow \mathbf{P}^n$ in the following way. For every point in \mathbf{P}^n we choose a homogeneous representation of the form $(w_0 : \dots : w_n)$ with $w_0 \in \{0, 1\}$ and set

$$P((w_0 : \dots : w_n), (\zeta_1, \dots, \zeta_n)) = \begin{cases} (1 : w_1 + \zeta_1 : \dots : w_n + \zeta_n), & \text{if } w_0 = 1, \\ (0 : w_1 : \dots : w_n) & \text{if } w_0 = 0. \end{cases}$$

It is easy to see that P is holomorphic in both variables and belongs to $\text{Lip}(1)$ with respect to each variable. So if $G \subset \mathbf{C}$ is a region, $f : G \rightarrow \mathbf{P}^n$ a holomorphic curve with Lipschitz constant L_1 , and $\phi : G \rightarrow \Delta$ is a holomorphic map with Lipschitz constant L_2 (Δ is equipped with the Euclidean metric), then $P(f, \phi) : G \rightarrow \mathbf{P}^n$ is a holomorphic curve with Lipschitz constant $L_1 + L_2$. We need two properties of P for future references:

$$\text{for every } w \text{ and every } \zeta \in \Delta : \quad P(P(w, \zeta), -\zeta) = w, \quad (31)$$

and

$$\text{dist}(P(w, \zeta'), P(w, \zeta'')) \leq \|\zeta' - \zeta''\|. \quad (32)$$

Lemma 6.2 *Let $E \subset \mathbf{C}$ be a K -sparse set. For every $z \in \mathbf{C}$ we denote by $s = s(z)$ the closest point from z in E . Then*

$$\sum_{m \in E \setminus \{s\}} \frac{1}{|z - m|^3} \leq 8K^{-3} \sum_{n=2}^{\infty} (\sqrt{n} - 1)^{-3} \leq 200K^{-3}.$$

and

$$\sum_{m \in E \setminus \{s\}} \frac{1}{|z - m|^4} \leq 16K^{-4} \sum_{n=2}^{\infty} (\sqrt{n} - 1)^{-4} \leq 800K^{-4}.$$

Proof. If we surround every point $m \in E$ by an open disc of radius $K/2$, centered at this point, these discs will be disjoint. We can assume that $z = 0$. Enumerate the points of E in the order of increase of their distances from the origin and let r_n be the n -th distance, $n = 1, 2, \dots$. Then n points belong to the closed disc of radius r_n and the discs of radii $K/2$, surrounding

these points are disjoint and all contained in $B(0, r_n + K/2)$. So we obtain $r_n \geq (\sqrt{n} - 1)K/2$. \square

Now we are ready to write the curve which will solve the interpolation problem. We apply Lemma 6.1 with $q = (1 : 0 : \dots : 0)$ in (c) and obtain the family of curves g , satisfying (a), (b) and (c) of Lemma 6.1. In our fixed coordinate system g has the following *meromorphic* homogeneous representation:

$$g(z, a) = (1 : g_1(z, a) : \dots : g_n(z, a)),$$

where $z \mapsto g_j$ are certain meromorphic functions in \mathbf{C} . (This representation is different from the one used in the proof of Lemma 6.1). Property (c) implies that $z \mapsto g_j(z)$ are actually holomorphic for $|z| > 3$ and using (23) we obtain

$$\|\psi^{-1}(g(z, a))\| \leq 7|z|^{-3} < 1/3, \quad |z| \geq 3 \quad (33)$$

and

$$\|\psi^{-1}(g(z, a')) - \psi^{-1}(g(z, a''))\| \leq 2\|a' - a''\| |z|^{-4}, \quad |z| \geq 3, \quad (34)$$

where ψ was defined in (22). Now we assume that a K -sparse set $E \subset \mathbf{C}$ is given with $K > 25$. An element $\mathbf{a} \in V^E$ is a function $E \rightarrow V$, $m \mapsto a_m$. We define

$$f(z, \mathbf{a}) = (1 : f_1(z, \mathbf{a}) : \dots : f_n(z, \mathbf{a})),$$

where

$$f_j(z, \mathbf{a}) = \sum_{m \in E} g_j(m - z, a_m), \quad 1 \leq j \leq n.$$

Let $s \in E$ be the closest point to z . Then we can single out the most important term in the previous sum:

$$f(z, \mathbf{a}) = P(g(s - z, a_s), \phi^s(z, \mathbf{a})), \quad (35)$$

where

$$\phi^s(z, \mathbf{a}) := \sum_{m \in E \setminus \{s\}} \psi^{-1}(g(m - z, a_m)),$$

and $\psi^{-1} \circ g$ is well defined in view of the estimate (33) and $|m - z| \geq K > 25$ for $m \in E \setminus \{s\}$. Now we derive from (33), (34) and Lemma 6.2 the following estimates:

$$\begin{aligned} \|\phi^s(z, \mathbf{a})\| &\leq \sum_{m \in E \setminus \{s\}} \|\psi^{-1}(g(m - z, a_m))\| \\ &\leq 7 \sum_{m \in E \setminus \{s\}} |m - z|^{-3} \leq 1400K^{-3} < 1/11 = \delta. \end{aligned} \quad (36)$$

and

$$\begin{aligned} \|\phi^s(z, \mathbf{a}') - \phi^s(z, \mathbf{a}'')\| &\leq 3\|\mathbf{a}' - \mathbf{a}''\|_\infty \sum_{m \in E \setminus \{s\}} |m - z|^{-4} \\ &\leq 1600K^{-4}\|\mathbf{a}' - \mathbf{a}''\|_\infty < \delta\|\mathbf{a}' - \mathbf{a}''\|_\infty, \end{aligned} \quad (37)$$

where $\|\mathbf{a}\|_\infty := \sup_{m \in E} \|a_m\|$. From (36) and Cauchy's estimate for derivatives follows that the spherical derivative of ϕ^s is uniformly bounded with respect to \mathbf{a} . Thus by Lemma 6.1 (a) and the Lipschitz property of P we conclude that the spherical derivative of f is uniformly bounded with respect to $\mathbf{a} \in V^E$.

It remains to show that for every $\mathbf{b} \in (\mathbf{P}^n)^E$ one can find $\mathbf{a} \in V^E$ such that $f(\cdot, \mathbf{a})|_E = \mathbf{b}$. We rewrite (35) for $z = s$ as

$$f_s(\mathbf{a}) := f(s, \mathbf{a}) = P(g(a_s), \phi_s(\mathbf{a})), \quad (38)$$

where $g(a) := g(0, a)$ and $\phi_s(\mathbf{a}) := \phi^s(s, \mathbf{a})$. From (36) follows $\phi_s(\mathbf{a}) \in \Delta := \{\zeta \in \mathbf{C}^n : \|\zeta\| < 1\}$, that is

$$\|\phi_s(\mathbf{a})\| < \delta, \quad \mathbf{a} \in V^E, \quad (39)$$

and from (37) we obtain

$$\|\phi_s(\mathbf{a}') - \phi_s(\mathbf{a}'')\| \leq \delta\|\mathbf{a}' - \mathbf{a}''\|_\infty, \quad \mathbf{a}', \mathbf{a}'' \in V^E. \quad (40)$$

The possibility of free interpolation is now obtained from the following version of the Inverse Function Theorem, where we put $S = \mathbf{P}^n$, $\epsilon = 1$, $L = 5$ and $\delta = 1/11$.

Lemma 6.3 *Let V and S be metric spaces, $W \subset V$ a subspace with the property (25). Let $g : V \rightarrow S$ be a map, such that $g|_W$ is surjective and in every ball $B(g(a), \delta)$, $a \in W$ there exists a right inverse to g , taking $g(a)$ to a and with Lipschitz constant L .*

Let E be arbitrary set, and $\mathbf{f} : V^E \rightarrow S^E$ be defined by

$$f_s(\mathbf{a}) := f(s, \mathbf{a}) = P(g(a_s), \phi_s(\mathbf{a})), \quad s \in E,$$

where $P : S \times \Delta \rightarrow S$ is a continuous map with properties (31) and (32) and $\phi_s : V^E \rightarrow \Delta$ satisfies (39) and (40). Assume that

$$L\delta < \epsilon/(1 + \epsilon). \quad (41)$$

Then \mathbf{f} is surjective.

Proof. Let $\mathbf{b} \in S^E$. Using surjectivity of $g|W$ we find $\mathbf{a}^0 \in W^E \subset V^E$ such that

$$g(a_m^0) = b_m, \quad m \in E.$$

Denote by g_m^{-1} the right inverses to g , defined in $B(b_m, \delta)$ and such that $g_m^{-1}(b_m) = a_m^0$. We construct inductively a sequence (\mathbf{a}^k) , $k = 1, 2, \dots$. Assume that \mathbf{a}^{k-1} is already defined. Then $\|\phi_m(\mathbf{a}^{k-1})\| < \delta$ for $m \in E$ by (39) and thus

$$\begin{aligned} & \text{dist} \left(P(b_m, -\phi_m(\mathbf{a}^{k-1})), b_m \right) \\ &= \text{dist} \left(P(b_m, -\phi_m(\mathbf{a}^{k-1})), P(b_m, 0) \right) \leq \delta, \end{aligned} \quad (42)$$

where we used (31) with $\zeta = 0$ and (32). So we can apply our right inverses g_m^{-1} to define \mathbf{a}^k from the conditions

$$g(a_m^k) = P(b_m, -\phi_m(\mathbf{a}^{k-1})), \quad m \in E. \quad (43)$$

Now we show by induction that

$$\|\mathbf{a}^k - \mathbf{a}^{k-1}\|_\infty \leq L^k \delta^k, \quad k = 1, 2, \dots \quad (44)$$

For $k = 1$ this follows from (42) and the Lipschitz property of g_m^{-1} . Now let $k \geq 2$. Then using (32) and (40) we obtain

$$\begin{aligned} & \text{dist} \left(P(b_m, -\phi_m(\mathbf{a}^{k-1})), P(b_m, -\phi_m(\mathbf{a}^{k-2})) \right) \\ & \leq \|\phi_m(\mathbf{a}^{k-1}) - \phi_m(\mathbf{a}^{k-2})\|_\infty \\ & \leq \delta \|\mathbf{a}^{k-1} - \mathbf{a}^{k-2}\|_\infty \leq \delta L^{k-1} \delta^{k-1} = \delta^k L^{k-1}. \end{aligned}$$

Applying Lipschitz property of g_m^{-1} we obtain (44). In view of (44) the sequence (\mathbf{a}^k) is a Cauchy sequence, and by (41) and (44) it remains within ϵ from its original point \mathbf{a}^0 , so by (25) it converges to some $\mathbf{a} \in V^E$. In view of (43) and continuity of P

$$g(a_m) = P(b_m, -\phi_m(\mathbf{a})), \quad m \in E.$$

Thus by (31) and (38)

$$f_m(\mathbf{a}) = P(g(a_m), \phi_m(\mathbf{a})) = P(P(b_m, -\phi_m(\mathbf{a})), \phi_m(\mathbf{a})) = b_m, \quad m \in E.$$

□

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